POLYNOMIAL BASES FOR COMPACT SETS IN THE PLANE

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1. **Introduction.** This work is essentially a generalization, in two directions, of theorems about the Taylor series expansion of functions analytic on the closed unit disk in the plane. In the first place, using work of J. L. Walsh, we replace the closed unit disk by any compact set E whose complement E' is connected, and regular in the sense that E' has a Green's function with pole at infinity. Such a set E will be called a *coregular set*. In the second place, as a generalization of work of B. Cannon and J. M. Whittaker, the Taylor series expansion is replaced by a large class of expansions of the form:

$$f(z) = \sum_{0}^{\infty} A_k P_k(z) \tag{1.1}$$

where (P_k) is a simple polynomial basis (that is, each P_k is a polynomial of exact degree k) which depends on E but not on f, each A_k is a constant depending on f, and the convergence is uniform on each compact subset of some open set O containing E. Such convergence will be called *compact-open on E* or *compact-open in O*.

The polynomial bases we study will be shown to give rise to maximal convergence as defined by Walsh. We shall state and prove, for such bases, exact analogs of Ostrowski's theorems concerning the relation between overconvergence of the Taylor series and lacunary structure of the Taylor series. Finally, the class of bases for which our results hold will be shown to include certain classical polynomial bases, such as the Faber polynomials and all kinds of orthogonal polynomials.

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2. **Some notation.** Let E be a coregular set and let F be the Green's function with pole at infinity for the complement E' of E. Let $K = \lim_{z\to\infty} |z| \exp(-F(z))$. K is called the *capacity* or *transfinite diameter* of E. We shall assume $K>0$. For each extended real number $x \geqq 1$ let E_x be the set of all z such that z is in E or $\exp(F(z)) < x$, and let L_x be the boundary of E_x.

Let (Z_n) be a sequence of points of E such that, if f is any function analytic on E and S_n is the interpolation polynomial for f in the points $Z_1, \ldots, Z_{n+1}$, then the sequence (S_n) converges to f uniformly on E. S_n is called the nth *interpolation*

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polynomial for f in the points (Z_n). Let $Q_0(z)=1$, $Q_n(z)=(z-Z_1)\cdots(z-Z_n)$. We shall call (Q_n) the *interpolation basis for* E in the points (Z_n).

The largest positive extended real number X such that f is analytic in E_X will be called the *modulus of analyticity of* f with respect to E. Note that $S_n(z)=\sum_0^n B_k Q_k(z)$ where

$$B_k = (1/2\pi i)\int_{L_y} (f(z)/Q_{k+1}(z))\,dz \tag{2.1}$$

for any y such that $1<y<X$. Thus $f(z)=\sum_0^\infty B_n Q_n$, where the convergence is compact-open in E_X, and

$$\limsup |B_n|^{1/n} = 1/KX. \tag{2.2}$$

$\sum_0^\infty B_n Q_n$ is called the *interpolation series for* f in the points (Z_n). See Walsh [9, pp. 65, 74, 170, 159, 157, 52–54] and [8, p. 606] for the above definitions and assertions.

Let $B=(B_n)$ and $Q=(Q_n)$ be sequences, and let $D=[D_{nk}]$ and $G=[G_{nk}]$ be infinite matrices. $D\cdot G$ will denote the matrix product of D and G whenever all the series thus defined converge. $B\cdot G$, $G\cdot Q$ and $B\cdot Q$ are defined by considering B as a matrix with one row and Q as a matrix with one column. $|B|$ will denote the sequence $(|B_n|)$ and $|D|$ will denote the matrix $[|D_{nk}|]$, while $N(B)$ will denote $\limsup |B_n|^{1/n}$. We will sometimes write $N(B_n)$ for $N(B)$.

3. **Effective polynomial bases.** For the remainder of the paper let $Q=(Q_n)$ be an interpolation basis belonging to the coregular set E, with capacity $K>0$. Let $P=(P_n)$ be any simple polynomial basis, and let $G=[G_{nk}]$ be the (unique) matrix of complex numbers such that

$$Q_n = \sum_{k=0}^{\infty} G_{nk}P_k \qquad (n = 0, 1, 2, \ldots). \tag{3.1}$$

Note that (3.1) can be written $Q=G\cdot P$, and that G is lower triangular. For each $x\geqq 1$ let $M(x)=(M_n(x))$, where

$$M_n(x) = (\max |P_n(z)|, z \text{ on } L_x) \qquad (n = 0, 1, 2, \ldots). \tag{3.2}$$

Let $R(x)=(R_n(x))$ be defined by

$$R_n(x) = \sum_{k=0}^{\infty} |G_{nk}|M_k(x) \qquad (n = 0, 1, 2, \ldots), \tag{3.3}$$

and define $I(x)=\limsup (R_n(x))^{1/n}$. Note that $R(x)=|G|\cdot M(x)$ and $I(x)=N(R(x))$.

By the maximum principle, $I(x)\leqq I(y)$ when $x\leqq y$. Moreover, $|Q_n(z)|\leqq R_n(x)$ for z on L_x, and since

$$\lim |Q_n(z)|^{1/n} = K\exp(F(z)) \tag{3.4}$$

where the convergence is compact-open in E' [9, p. 159], we have $I(x)\geqq Kx$ for

$x>1$. The fact that $I(1)\geqq K$ follows from the lemma in [9, p. 77], applied to the $(Q_n(z))$ on L_1.

LEMMA. *$I(x)$ is independent of Q.*

Proof. Let $q=(q_n)$ be any other interpolation basis for E. Let $g=[g_{nk}]$ be the matrix such that $q=g\cdot P$, and let $r(x)=|g|\cdot M(x)$. We wish to show that $N(r(x))=N(R(x))$. Let D be the matrix such that $Q=D\cdot q$. Note that D is lower triangular and $D_{nn}=1$ for all n. Also, since D and g are lower triangular and P is a basis, we have $G\cdot P=Q=D\cdot q=D\cdot(g\cdot P)=(D\cdot g)\cdot P$, so $G=D\cdot g$.

We consider first the case $K=1$. Let y be any number greater than 1. From (2.1) we get

$$D_{nk} = (1/2\pi i)\int_{L_y} (Q_n(z)/q_{k+1}(z))\,dz$$

so that

$$|D_{nk}| \leqq c(\max|Q_n(z)|,\ z \text{ on } L_y)/(\min|q_{k+1}(z)|,\ z \text{ on } L_y)$$

where c is a positive constant. Applying (3.4) with $K=1$ to Q and q, we get

$$\liminf_n (\min(\min|q_{k+1}(z)|,\ z \text{ on } L_y),\ k=0,1,\dots,n)^{1/n} \geqq 1$$

and

$$\limsup_n (\max|D_{nk}|,\ k=0,1,\dots,n)^{1/n} \leqq y.$$

Since this is so for all $y>1$, and since $|D_{nn}|=1$ for all n, we get

$$\limsup_n (\max|D_{nk}|,\ k=0,1,\dots,n)^{1/n} = 1. \tag{3.5}$$

Now $N(R(x))=N(|G|\cdot M(x))=N(|D\cdot g|\cdot M(x))\leqq N(|D|\cdot|g|\cdot M(x))=N(|D|\cdot r(x))$. But $N(|D|\cdot r(x))\leqq N(r(x))$ because of (3.5) and the fact that $N(r(x))\geqq Kx=x\geqq 1$. Reversing the roles of r and R we get $N(r(x))\leqq N(R(x))$ and so $N(r(x))=N(R(x))$. This completes the proof of the lemma for the case $K=1$. The proof for the general case follows from a consideration of the polynomial basis $(P_n(Kz))$, on the set $(1/K)E=$ the set of all z such that Kz is in E, since $(1/K)E$ has capacity 1. The corresponding interpolation bases are $(Q_n(Kz)/K^n)$ and $(q_n(Kz)/K^n)$. This completes the proof of the lemma.

DEFINITION. P is called *effective* on Cl $(E_x)=$ the closure of E_x if and only if, for some interpolation basis Q, any function f analytic on Cl (E_x) can be expanded in the form (1.1), the convergence being uniform on Cl (E_x), with $A=(A_k)$ defined by

$$A_k = \sum_{n=0}^{\infty} B_n G_{nk} \qquad (k=0,1,2,\dots), \tag{3.6}$$

where B is defined by (2.1) for some $y>1$, and G is defined by (3.1). Note that (3.6) can be written $A=B\cdot G$.

THEOREM 1. *P is effective on* Cl (E_x) *if and only if $I(x)=Kx$.*

Proof. Let $I(x)=Kx$ for some $x \geqq 1$ and let f be analytic on $\mathrm{Cl}\,(E_x)$. Let $f=B \cdot Q$ for some interpolation basis Q, define G by (3.1), and define A by (3.6). From (2.2) it follows that $N(B)<1/Kx$, while $N(R(x))=Kx$, so $|B| \cdot R(x)$ converges absolutely. Thus $f=B \cdot Q=B \cdot (G \cdot P)=(B \cdot G) \cdot P=A \cdot P$, where the absolute convergence of $|B| \cdot R(x)$ implies the convergence of every term in $B \cdot G$ and the uniform convergence of $A \cdot P$ on $\mathrm{Cl}\,(E_x)$, as well as the equality $B \cdot (G \cdot P)=(B \cdot G) \cdot P$.

Conversely, let $I(x) \neq Kx$ and let X be any number such that $Kx<KX<I(x)$. We construct a function f, with modulus of analyticity X with respect to E, such that f cannot be expanded in the necessary way. Let $C_k=(G_{nk})$ be the kth column of G, and let $B=(B_n)=(1/(KX)^n)$.

Case 1. Suppose that for some k, $B \cdot |C_k|$ does not converge. Choose real numbers (y_n) such that $G_{nk}=|G_{nk}| \exp{(iy_n)}$ and let $b=(b_n)=(\exp{(-iy_n)}/(KX)^n)$. Define $f=b \cdot Q$. Then from (2.2) f has modulus of analyticity X, but f cannot be expanded in the necessary way, since if it could, then $A_k=b \cdot C_k=B \cdot |C_k|$, which is a contradiction.

Case 2. Suppose $B \cdot |C_k|$ converges for all k. If M is defined by (3.2) then

$$KX < I(x) \leqq \limsup_n\, (\max |G_{nk}|M_k(x),\, k = 0, 1, \ldots, n)^{1/n}$$

and so there exist sequences $n(j)$ and $k(j)$ of positive integers, with $n(j)$ strictly increasing, such that

$$|G_{n(j)k(j)}|M_{k(j)}(x) > (KX)^{n(j)}. \tag{3.7}$$

Now the terms $G_{n(j)k(j)}$ cannot all belong to a finite number of the C_k, since if infinitely many belong to C_k for some k, then $B \cdot |C_k|$ would not converge. Consequently we may assume that $k(j)$ is also strictly increasing. Choose three sequences $u(j)$, $v(j)$ and $w(j)$ of positive integers in the following way: let $u(1)=n(1)$, $v(1)=k(1)$. Choose $w(1)>u(1)$ such that

$$\left| \sum_{h=w(1)}^{\infty} G_{hv(1)}/(KX)^h \right| < (1/2)|G_{u(1)v(1)}|/(KX)^{u(1)}.$$

If $u(j)$, $v(j)$ and $w(j)$ have been chosen, choose $u(j+1)$ and $v(j+1)$ such that $u(j+1)=n(r)$, $v(j+1)=k(r)$ for some r, $u(j+1)>w(j)$, and $G_{nv(j+1)}=0$ for $n<w(j)$. This last requirement can be met by taking $v(j+1) \geqq w(j)$. Finally, choose $w(j+1)>u(j+1)$ such that

$$\left| \sum_{h=w(j+1)}^{\infty} G_{hv(j+1)}/(KX)^h \right| < (1/2)|G_{u(j+1)v(j+1)}|/(KX)^{u(j+1)}.$$

Let $b=(b_n)$ where $b_n=1/(KX)^n$ when $n=u(j)$ for some j, and $b_n=0$ otherwise. Define $f=b \cdot Q$. From (2.2) f has modulus of analyticity X. Now for all j,

$$\begin{aligned} |A_{v(j)}| = |b \cdot C_{v(j)}| &= \left| G_{u(j)v(j)}/(KX)^{u(j)} + \sum_{h=w(j)}^{\infty} b_h G_{hv(j)} \right| \\ &> (1/2)|G_{u(j)v(j)}|/(KX)^{u(j)}. \end{aligned}$$

Thus

$$|A_{v(j)}|M_{v(j)}(x) > (1/2)|G_{u(j)v(j)}|M_{v(j)}(x)/(KX)^{u(j)} > 1/2$$

by (3.7). Thus $A \cdot P$ does not converge uniformly on L_x. This completes the proof of the theorem.

Because of Theorem 1, the function I, defined for all $x \geqq 1$, is called the *index of effectivity* of P on E. By [9, p. 77], $M_n(y) \leqq (y/x)^n M_n(x)$ whenever $x<y$, and so $I(y) \leqq (y/x)I(x)$. Together with the fact that $I(x) \leqq I(y)$ whenever $x<y$, we conclude that either $I(x)=\infty$ for all $x \geqq 1$ or else I is finite-valued and continuous for all $x \geqq 1$. Moreover, suppose P is effective on $\mathrm{Cl}\,(E_x)$ for some x and let y be any number $>x$. Then $I(y) \leqq (y/x)I(x)=(y/x)Kx=Ky$, and since $I(y) \geqq Ky$, we get $I(y)=Ky$, so P is effective on $\mathrm{Cl}\,(E_y)$.

Because of the lemma, we know that the effectivity of P does not depend on the interpolation basis Q. The coefficients A in (1.1) also do not depend on Q when P is effective, a consequence of

THEOREM 2. *Let P be a simple polynomial basis, and let $f(z)=\sum_0^\infty A_n P_n(z)$, where the convergence is absolute and compact-open on an open set containing* $\mathrm{Cl}\,(E_x)$ *for some $x \geqq 1$. Let G be defined by* (3.1) *for any interpolation basis Q, and let C_j be the jth column of G. If $N(C_j) \leqq Kx$ for some j, then $A_j=B \cdot C_j$, where B is defined by* (2.1).

Proof. Let $A=(A_n)$, and let y be any number $>x$ such that $A \cdot P$ converges absolutely and uniformly on L_y. From (2.2), $N(B) \leqq 1/Ky < 1/Kx$, while $N(C_j) \leqq Kx$ so $B \cdot C_j$ converges. Let $D=[D_{nk}]$ be the matrix inverse of G (G is lower triangular with nonzero diagonal elements). Then $P=D \cdot Q$ since D and G are lower triangular, and from (2.1) we conclude that D is given by:

$$D_{nk} = (1/2\pi i) \int_{L_y} (P_n(z)/Q_{k+1}(z))\, dz. \tag{3.8}$$

Again from (2.1) we get for each k:

$$\begin{aligned} B_k &= (1/2\pi i) \int_{L_y} (f(z)/Q_{k+1}(z))\, dz \\ &= \sum_{n=0}^{\infty} A_n (1/2\pi i) \int_{L_y} (P_n(z)/Q_{k+1}(z))\, dz = \sum_{n=0}^{\infty} A_n D_{nk}. \end{aligned}$$

Thus $B=A \cdot D$. Since $A \cdot P$ converges absolutely and uniformly on L_y, the function H defined by: $H(z)=\sum_{n=0}^\infty |A_n P_n(z)|$ is continuous on L_y. Therefore from (3.8), the kth term of $|A| \cdot |D|$ satisfies the following:

$$\begin{aligned} \sum_{n=0}^{\infty} |A_n|\, |D_{nk}| &\leqq 1/(2\pi) \sum_{n=0}^{\infty} |A_n| \int_{L_y} |P_n(z)/Q_{k+1}(z)|\, |dz| \\ &= 1/(2\pi) \int_{L_y} (H(z)/|Q_{k+1}(z)|)\, |dz| \leqq J/(\min |Q_{k+1}(z)|,\, z \text{ on } L_y), \end{aligned}$$

where J is a positive constant. From (3.4) we conclude that $N(|A| \cdot |D|) \leqq 1/Ky$, so $(|A| \cdot |D|) \cdot |C_j|$ converges. Therefore, $B \cdot C_j=(A \cdot D) \cdot C_j=A \cdot (D \cdot C_j)=A_j$. This completes the proof of the theorem.

COROLLARY. *Let P be effective on* Cl (E_x) *for some* $x \geqq 1$ *and let f be analytic on* Cl (E_x). *Then the coefficients A in* (1.1) *are unique.*

Proof. Let $f = B \cdot Q$ for any interpolation basis Q, and let X be the modulus of analyticity of f. From (2.2), $N(B) = 1/KX$, and so, if $R(x)$ is defined by (3.3), then from the first part of the proof of Theorem 1 we see that $|B| \cdot R(y)$ converges for all y such that $N(R(y)) < KX$. But from the remarks following Theorem 1, $N(R(y)) = Ky$ for all $y \geqq x$, so the convergence of $|B| \cdot R(y)$ is compact-open on $1 \leqq y < X$. Thus the convergence of $A \cdot P$ is absolute and compact-open in E_x. If G is defined by (3.1) then $|G_{nj}| \leqq R_n(x)/M_j$, so that if C_j is the jth column of G then $N(C_j) \leqq N(R(x)) = I(x) = Kx$. This completes the proof of the corollary.

See Whittaker [10, pp. 5–12, 18–20, 60–62] for the definitions and results in this section (except the lemma) for the special case E = the closed unit disk, $Q_n(z) = z^n$ for all n. Whittaker also treats effectivity in E_x for this special case, and many of those results can also be generalized.

4. **Maximal convergence.** Suppose f is analytic on E with modulus of analyticity X. For each n let S_n be a polynomial of degree n, and let $m = (m_n)$ where $m_n = (\max |f(z) - S_n(z)|, z \text{ on } E)$. If $N(m) \leqq 1/X$ then (S_n) is said to *converge maximally* to f on E [9, pp. 79, 80].

THEOREM 3. *If P is effective on E then P always gives rise to maximal convergence on E.*

Proof. Let f be analytic on E with modulus of analyticity X. Then $f = A \cdot P$, where $A = B \cdot G$, B being defined by (2.1), and G by (3.1) for any interpolation basis Q. Now if x is any number such that $1 \leqq x < X$ then, from the first part of the proof of Theorem 1, $A \cdot P = (B \cdot G) \cdot P = B \cdot (G \cdot P) = B \cdot Q$ on L_x and therefore on E. If $M(x)$ and $R(x)$ are defined by (3.2) and (3.3), then for z on L_x we have:

$$\left| f(z) - \sum_{k=0}^{n} A_k P_k(z) \right| = \left| \sum_{k=n+1}^{\infty} A_k P_k(z) \right| \leqq \sum_{k=n+1}^{\infty} |A_k| \, |P_k(z)|$$

$$\leqq \sum_{k=n+1}^{\infty} \left(\sum_{j=k}^{\infty} |B_j| \, |G_{jk}| \right) M_k(x) \leqq \sum_{j=n+1}^{\infty} |B_j| R_j(x)$$

since G is lower triangular. From (2.2) and (3.4) we get:

$$\limsup_{n} \left(\sum_{j=n+1}^{\infty} |B_j| R_j(x) \right)^{1/n} \leqq x/X.$$

Letting $x = 1$ completes the proof of the theorem.

It is not true that every simple polynomial basis which gives rise to maximal convergence is effective. For example, let E be the closed unit disk, which has capacity $K = 1$. Let $Q_n(z) = z^n$ for all n. Let x be any number > 1. Let $P_0(z) = 1$, $P_n(z) = (z - x) z^{n-1}$ for all $n \geqq 1$. If G is defined by (3.1) then $G_{nk} = x^{n-k}$, so

$I(1) \geqq x > 1$, and by Theorem 1, P is not effective on E. On the other hand, any function analytic on E can be expanded in the form (1.1), with maximal convergence [9, p. 159].

The effectivity of a polynomial basis P on Cl (E_x) for some coregular set E and some $x \geqq 1$ does not imply anything about the maximum values of $|P_n(z)|$ on L_x. For, suppose P is any polynomial basis and let (T_k) be any sequence of nonzero complex numbers. Let $p=(p_k)=(T_k P_k)$ and let $g=[g_{nk}]$ be defined by $Q=g \cdot p$, where Q is any interpolation basis. For any $x \geqq 1$ let $M(x)$ and $R(x)$ be defined by (3.2) and (3.3), let $m(x)=(m_n(x))$ be defined by

$$m_n(x) = (\max |p_n(z)|, z \text{ on } L_x) = |T_n|(\max |P_n(z)|, z \text{ on } L_x) = |T_n| M_n(x),$$

and let $r(x)=|g| \cdot m(x)$. Now if $Q=G \cdot P$ then $g_{nk}=G_{nk}/T_k$ and so $r(x)=|g| \cdot m(x) = |G| \cdot M(x) = R(x)$. Thus the values of the index of effectivity are unchanged if the P_n are multiplied by nonzero constants, and so, from Theorem 1, effectivity is unaffected.

We may therefore assume, when it is convenient, that P is made up of monic polynomials. On the other hand, with this assumption we have $N(M(x))=Kx$. In fact, we even have:

THEOREM 4. *Let P be effective on* Cl (E_x) *for some* $x \geqq 1$, *and suppose every* P_n *is monic. Then* $\lim (M_n(x))^{1/n}=Kx$.

Proof. Let D be defined by $P=D \cdot Q$ for any interpolation basis Q. Then D is given by (3.8) for any $y>1$. Moreover, $D_{nn}=1$ for all n, since P_n and Q_n are monic. Let $(M_{k(n)}(y))$ be any subsequence of $M(y)$. Then from (3.8):

$$1 \leqq c(\max |P_{k(n)}(z)|, z \text{ on } L_y)/(\min |Q_{k(n)+1}(z)|, z \text{ on } L_y)$$

where c is a positive constant. From (3.4) we get $Ky \leqq \limsup_n (M_{k(n)}(y))^{1/k(n)}$. On the other hand, since $D_{nn}=1$ for all n, $M_{k(n)}(y) \leqq R_{k(n)}(y)$, so

$$Ky \leqq \limsup_n (M_{k(n)}(y))^{1/k(n)} \leqq I(y), \tag{3.9}$$

for all $y>1$. Now if $x>1$ then (3.9) and Theorem 1 show that

$$\limsup_n (M_{k(n)}(x))^{1/k(n)} = Kx$$

for every subsequence of $M(x)$. Therefore the theorem is proved for $x>1$. If $x=1$ then (3.9) shows that $\limsup_n (M_{k(n)}(1))^{1/k(n)} \leqq Ky$ for all $y>1$ so

$$\limsup_n (M_{k(n)}(1))^{1/k(n)} \leqq K,$$

while [9, p. 77] and our theorem for $x>1$ show that $\limsup_n (M_{k(n)}(1))^{1/k(n)} \geqq K$, so the theorem is proved for $x=1$. This completes the proof of the theorem.

COROLLARY. *Let P be effective on E and suppose every P_n is monic. Let f be analytic on E, with modulus of analyticity X. If $A=(A_n)$ is defined by* (1.1) *then*

$N(A)=1/KX$. Moreover, if O is any bounded continuum, not a single point, in E', then

$$\lim\,(\max|P_n(z)|,\ z \text{ in } O)^{1/n} = K\max(\exp(F(z)),\ z \text{ in } O).$$

Proof. Since $A\cdot P$ converges maximally on E, $N(A_nM_n(X))=1$ [8, p. 606]. Thus $N(A)=1/KX$. Now let $x=(\max\exp(F(z)), z \text{ in } O)$, and let $m_n=(\max|P_n(z)|, z \text{ in } O)$. From the maximum principle and the theorem, $N(m_n)\leqq Kx$. Suppose there exists a subsequence $(m_{k(n)})$ such that $\lim_n (m_{k(n)})^{1/k(n)}<Kx$. Let $a_j=1/(Kx)^j$ when $j=k(n)$, $a_j=0$ otherwise. From the theorem, $N(a_jM_j(y))=y/x$ for all $y\geqq 1$, so $a\cdot P$ converges maximally to a function with modulus of analyticity x. But $N(a_jm_j)<1$, which contradicts [8, p. 606]. This completes the proof of the corollary.

5. **Overconvergence.** Suppose f is analytic on E with modulus of analyticity X. Let P be any simple polynomial basis effective on E, and let $S_n=\sum_{j=0}^n A_jP_j$, where $A=(A_j)$ is defined by (1.1). From the first part of the proof of Theorem 1, (S_n) converges uniformly on any compact subset of E_X. Now (S_n) cannot converge uniformly on any neighborhood of a point of L_X [9, p. 83], but it may happen that for some strictly increasing sequence $n(k)$ of positive integers, $(S_{n(k)})$ converges uniformly on such a neighborhood. When this happens we shall say that $(S_{n(k)})$ *overconverges* on the neighborhood. (Such overconvergence does occur. For instance, if (O_n) is a sequence of mutually disjoint, simply connected regions such that O_1 contains E_x but not $\mathrm{Cl}\,(E_x)$ for some $x>1$, then there exists a function f, analytic in each O_n, with modulus of analyticity x with respect to E, and a simple polynomial basis P, effective on E, such that a subsequence $(S_{n(k)})$ of (S_n) converges to f uniformly on every compact subset of every O_n, but does not converge uniformly on any region properly containing some O_n. See Bourion [1, pp. 268–270] for the proof in the special case $E_x=$ the unit disk, $P_n(z)=z^n$. The proof can be generalized, with $P=$ any interpolation basis belonging to E.)

Let f be analytic on E with modulus of analyticity X, and let P be any simple polynomial basis effective on E. Let Y be some positive number <1 and let $n(k)$ be a strictly increasing sequence of positive integers. Let H be the set of all integers j such that $Yn(k)\leqq j\leqq n(k)$ for some k. If $\sum_{j\in H} A_jP_j$ converges compact-openly in E_x for some $x>X$, we shall say that $A\cdot P$ is of *lacunary structure* with respect to $n(k)$, since $A\cdot P$ is then the sum of a series with gaps and a series with a larger modulus of analyticity than f.

THEOREM 5. *Let f be analytic on E with modulus of analyticity X, and let P be effective on E. If the sequence (S_n) of partial sums of* (1.1) *has a subsequence $(S_{n(k)})$ which overconverges on some neighborhood of a point of L_X, then* (1.1) *is of lacunary structure with respect to $n(k)$.*

Proof. Let Q be any interpolation basis for E and let G be defined by (3.1). For fixed n, $S_n=\sum_{j=0}^n A_jP_j=\sum_{h=0}^n b_{hn}Q_h$ where, by (2.1),

$$b_{hn} = 1/(2\pi i)\int_{L_y} (S_n(z)/Q_{h+1}(z))\,dz$$

for any $y>1$. Choose y such that $y>X$ and the distance from L_y to E is >1. Since $Q=G\cdot P$ and P is a basis,

$$A_j = 1/(2\pi i)\sum_{h=j}^{n} G_{hj}\int_{L_y}(S_n(z)/Q_{h+1}(z))\,dz$$

for all $j\leqq n$. Thus if J is the length of L_y divided by 2π, then

$$|A_j| \leqq \frac{J(\max|S_n(z)|,\ z \text{ on } L_y)}{(\min|Q_{j+1}(z)|,\ z \text{ on } L_y)}\sum_{h=j}^{n}|G_{hj}| \tag{5.1}$$

for all $j\leqq n$, since $|Q_{j+1}(z)|\leqq|Q_{k+1}(z)|$ for all $j\leqq k$ and all z on L_y. As was pointed out immediately before Theorem 4, we may assume that every P_j is monic. Since $(S_{n(k)})$ overconverges, we have

$$\limsup_k\,(\max|S_{n(k)}(z)|,\ z \text{ on } L_y)^{1/n(k)} < y/X \tag{5.2}$$

[7, p. 201] and [9, pp. 67 and 77]. Choose Y such that $0<Y<1$ and

$$\limsup_k\,(\max|S_{n(k)}(z)|,\ z \text{ on } L_y)^{1/Yn(k)} < y/X. \tag{5.3}$$

Let H be the set of all integers j such that $Yn(k)\leqq j\leqq n(k)$ for some k. We first examine the case $K=1$. Let $M(1)$ and $R(1)$ be defined by (3.2) and (3.3). Now $R_h(1)\geqq|G_{hj}|M_j(1)$ for all h and j, so (in the following six summations let k be the smallest integer such that $j\leqq n(k)$)

$$\limsup_{j\in H}\left(\sum_{h=j}^{n(k)}|G_{hj}|\right)^{1/j} \leqq \limsup_{j\in H}\left(\sum_{h=j}^{n(k)}R_h(1)/M_j(1)\right)^{1/j}.$$

From Theorem 4, $\limsup_{j\in H}(M_j(1))^{-1/j}=1$ so

$$\limsup_{j\in H}\left(\sum_{h=j}^{n(k)}|G_{hj}|\right)^{1/j} \leqq \limsup_{j\in H}\left(\left(\sum_{h=j}^{n(k)}R_h(1)\right)^{1/n(k)}\right)^{n(k)/j}.$$

But $\limsup(R_h(1))^{1/h}=1$ by Theorem 1, so

$$\limsup_{j\in H}\left(\sum_{h=j}^{n(k)}R_h(1)\right)^{1/n(k)} = 1,$$

and since $1\leqq n(k)/j\leqq 1/Y$, we get

$$\limsup_{j\in H}\left(\sum_{h=j}^{n(k)}|G_{hj}|\right)^{1/j} \leqq 1.$$

Combining this result with (5.1), (5.2), (5.3) and (3.4) we get $\limsup_{j\in H}|A_j|^{1/j}<1/X$. Then by Theorem 4, $\sum_{j\in H}A_jP_j$ converges compact-openly in E_x where $x=1/\limsup_{j\in H}(|A_j|)^{1/j}>X$. This completes the proof of the theorem for the case $K=1$. The proof for the general case follows from a consideration of $(P_j(Kz))$ on $(1/K)E$. This completes the proof of the theorem. See [6, p. 185] for the theorem in the case $E=$ the closed unit disk, $P_n(z)=z^n$.

The converse theorem, for regular points of f, was also proved by Ostrowski for the case $E=$the closed unit disk, $P_n(z)=z^n$ [5]. This converse theorem was generalized to include not only series, but also sequences of lacunary structure converging to functions analytic on coregular sets, by I. Mosesson in his unpublished Harvard thesis [4]. The following converse to Theorem 5 is therefore a special case of Mosesson's generalization. It also follows immediately from more general work of Walsh.

THEOREM 6. *Let f be analytic on E with modulus of analyticity X. Let P be effective on E and suppose* (1.1) *is of lacunary structure with respect to $n(k)$. Then $(S_{n(k)})$ converges uniformly in a neighborhood of every regular point of f on L_X.*

Proof. Let Y be a positive number <1 such that $\sum_{j\in H} A_jP_j$ converges compact-openly in E_x for some $x>X$, where H is the set of all integers j such that $Yn(k) \leqq j \leqq n(k)$ for some k. For each k, let $h(k)$ be the greatest integer $<Yn(k)$. Let Z_0 be any regular point of f on L_X. We may assume $A_j=0$ for all j in H. Thus $S_{h(k)}=S_{n(k)}$ for all k. Let O be any compact continuum, not a single point, containing Z_0 and contained in the complement of E_X, and let $x=(\max\exp(F(z)), z \text{ in } O)$. Since (S_n) converges maximally on E,

$$\limsup\,(\max |S_n(z)|, z \text{ in } O)^{1/n} \leqq x/X$$

[8, p. 607]. Thus

$$\begin{aligned}\limsup_k\,&(\max |S_{n(k)}(z)|, z \text{ in } O)^{1/n(k)}\\ &= \limsup_k\,(\max |S_{h(k)}(z)|, z \text{ in } O)^{1/n(k)}\\ &= \limsup_k\,((\max |S_{h(k)}(z)|, z \text{ in } O)^{1/h(k)})^{h(k)/n(k)} \leqq (x/X)^Y < x/X.\end{aligned}$$

The theorem now follows from [7, p. 201].

COROLLARY *Let P be effective on E and let f be analytic on E with modulus of analyticity X. Let (A_n) be defined by* (1.1) *and let $n(k)$ be a sequence of positive integers such that, for some positive number $Y<1$, $n(k)<Yn(k+1)$. If $A_n=0$ for all $n\neq n(k)$, then L_X is the natural boundary of f.*

Proof. $A\cdot P$ cannot converge uniformly in any neighborhood of a point of L_X ([9, p. 83], see [4]). This completes the proof of the corollary.

See Bourion [2, pp. 6–13] for the definitions and theorems in this section, for the case $E=$the closed unit disk, $P_n(z)=z^n$.

6. **Examples.** Let E be the coregular set of all points on and within a Jordan curve and let P be a simple polynomial basis which is orthonormal with respect to integration over E (or around the boundary of E), with any positive continuous function $W(z)$ allowed as weight function. If f is any function analytic on E with modulus of analyticity X, then f can be expanded in the form (1.1), where

$$A_k = \int W(z)f(z)P_k(z)'\,dz \tag{6.1}$$

(where $P_k(z)'$ is the complex conjugate of $P_k(z)$), the convergence being absolute and compact-open in E_X [9, pp. 91–97 and 128–130]. Let Q be any interpolation basis for E and let G be defined by (3.1). If C_k is the kth column of G then, by (6.1),

$$|G_{nk}| \leqq J(\max W(z), z \text{ on } E)(\max |P_k(z)|, z \text{ on } E)(\max |Q_n(z)|, z \text{ on } E),$$

where J is the area of E (or the length of L_1). Thus by (3.4),

$$N(C_k) \leqq \limsup_n (\max |Q_n(z)|, z \text{ on } E)^{1/n} = K$$

and so, by Theorem 2, P is effective on E.

The same proof applies to polynomial bases orthonormal over more general coregular sets, with more general weight functions. Almost exactly the same proof applies to the Faber polynomials belonging to an analytic Jordan curve. See [3, pp. 86–87] for a definition of the Faber polynomials.

The zeros of an effective polynomial basis need not be very well behaved. In fact, if (t_n) is any sequence of complex numbers, and E is any coregular set, then there exists a simple polynomial basis P, effective on E, such that each t_n is a zero of infinitely many of the P_k. To prove this, let (T_n) be the sequence $(t_1, t_1, t_2, t_1, t_2, t_3, \ldots)$ and choose a sequence $h(n)$ of positive integers such that $h(n+1) \geqq h(n)+2$ and $\limsup_n |T_n|^{1/h(n)} \leqq 1$. Let Q be any interpolation basis belonging to E, with $Q_n(z) = (z-Z_1)\cdots(z-Z_n)$. For $k = h(n)$ let $P_k(z) = (z-T_n)Q_{k-1}(z)$, while for $k \neq h(n)$ let $P_k(z) = Q_k(z)$. If G is defined by (3.1) then $G_{jj} = 1$ for all j, and $G_{jk} = 0$ for $j \neq k$ except when $j = h(n)$ and $k = h(n)-1$, in which case $G_{jk} = T_n - Z_{h(n)}$. It is not hard to see directly from definition (3.3), and (3.4), that P is effective on E.

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